Appendix: Derivation of the Distribution Truncation in EKF manner

The problem is to calculate the new mean $q_k$ and covariance $Q_k$ of the region which satisfies an inequality constraint $\varphi^T x \leq b_k$ under the prior normal distribution with the mean $q_{k-1}$ and covariance $Q_{k-1}$. It is reduced to a simpler case that the prior mean is 0, the prior covariance is $I$ and the constraint is $x'_1 \leq c_k$, by applying the following linear transform:

$$x' = RW^{-\frac{1}{2}}T(x - q_{k-1}) \tag{1}$$

where $R$ and $T$ are orthogonal matrices and $W$ is diagonal matrix respectively, and they satisfy the following equations:

$$TWT^T = Q_{k-1} \tag{2}$$

$$RW^{-\frac{1}{2}}T\varphi_k = ((\varphi_k^T Q_{k-1}\varphi_k)^{\frac{1}{2}}, 0, \ldots, 0)^T \tag{3}$$

$$c_k = \frac{b_k - \varphi_k^T q_{k-1}}{(\varphi_k^T Q_{k-1}\varphi_k)^{\frac{1}{2}}} \tag{4}$$

Since the transformed distribution is a isotropic (note that its covariance is an identity matrix), the components are independent to each other and the distribution can be represented as the product of the marginal distributions.

$$p(x') = \prod_i p_i(x'_i) = \prod_i \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x'^2_i}{2}\right) \tag{5}$$

Here, the inequality constraint is $x'_1 \leq c_k$, namely the probability density in the region $x'_1 > c_k$ should be zero by the distribution truncation. Therefore, the original distribution should be normalized so that the definite integral of the normalized distribution over $x'_1 \leq c_k$ is 1.

$$\int_{-\infty}^{c_k} \cdots \int_{-\infty}^{c_k} p(x')dx'_1 \cdots dx'_n = \int_{-\infty}^{c_k} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x'^2_1}{2}\right)dx'_1 = \frac{1}{2} \left\{ 1 + \text{erf}\left(\frac{c_k}{\sqrt{2}}\right) \right\} \tag{6}$$

where erf(·) denotes the error function:

$$\text{erf}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2})dt. \tag{7}$$

This function is not an analytical function but it can be quite rapidly calculated using a common numerical calculation package. The normalized distribution is as follows:

$$p'(x') = \alpha \prod_i p_i(x'_i) \tag{8}$$

$$\alpha \equiv \frac{\sqrt{2}}{\sqrt{\pi}(1 + \text{erf}(c_k/\sqrt{2}))} \tag{9}$$

The mean and covariance of the above distribution are obtained as follows, using the fact that each $x'_i$ is independent:

$$\mu_1 = E[x'_1] = \alpha \int_{-\infty}^{c_k} x'_1 \exp\left(-\frac{x'^2_1}{2}\right)dx'_1 = -\alpha \exp\left(-\frac{c_k^2}{2}\right) \tag{10}$$
\[ \mu_i = E[x'_i] = 0 \quad (i \neq 1) \quad (11) \]
\[ \sigma^2_{11} = E[x'_1 - \mu_1][x'_1 - \mu_1] = \alpha \int_{-\infty}^c (x'_1 - \mu_1)^2 \exp(-\frac{c^2}{2}) dx'_1 \]
\[ = 1 - \alpha c \exp(-\frac{c^2}{2}) - \mu^2_1 \quad (12) \]
\[ \sigma^2_{ii} = E[x'_i - \mu_i][x'_i - \mu_i] = 1 \quad (i \neq 1) \quad (13) \]
\[ \sigma^2_{ij} = E[x'_i - \mu_i][x'_j - \mu_j] = 0 \quad (i \neq j). \quad (14) \]

By rewritten the above equations introducing a variable
\[ \nu_k \equiv \mu_1 \quad (15) \]
\[ = -\sqrt{\frac{2}{\pi}} \exp(-\frac{c^2}{2})/(1 + \text{erf}(-\frac{c_\nu}{\sqrt{2}})), \quad (16) \]

the mean and covariance are
\[ \mu_k = (\nu_k, 0, \cdots, 0) \quad (17) \]
\[ S_k = \text{diag}\{1 + c_k \nu_k - \nu^2_k, 1, \cdots, 1\} \quad (18) \]

where \text{diag}\{a, b, \cdots\} represents a diagonal matrix whose diagonal elements are \(a, b, \cdots\). Then the truncated mean and variance are expressed as
\[ q_k = TW^k R^T \mu_k + q_{k-1} \quad (19) \]
\[ Q_k = TW^k R^T S_k RW^k T^T \quad (20) \]

by applying inversely the transform Eq.1.