# Appendix: Derivation of the Distribution Truncation in EKF manner

The problem is to calculate the new mean $\boldsymbol{q}_k$ and covariance $\boldsymbol{Q}_k$ of the region which satisfies an inequality constraint $\varphi_k^T \boldsymbol{x} \leq b_k$ under the prior normal distribution with the mean $\boldsymbol{q}_{k-1}$ and covariance $\boldsymbol{Q}_{k-1}$. It is reduced to a simpler case that the prior mean is $\boldsymbol{0}$, the prior covariance is $\boldsymbol{I}$ and the constraint is $x'_1 \leq c_k$, by applying the following linear transform:

$$\boldsymbol{x'} = \boldsymbol{R}\boldsymbol{W}^{-\frac{1}{2}}\boldsymbol{T}^T(\boldsymbol{x} - \boldsymbol{q}_{k-1}) \tag{1}$$

where $\boldsymbol{R}$ and $\boldsymbol{T}$ are othogonal matrices and $\boldsymbol{W}$ is diagonal matrix respectively, and they satisfy the following equations:

$$\boldsymbol{T}\boldsymbol{W}\boldsymbol{T}^T = \boldsymbol{Q}_{k-1} \tag{2}$$

$$\boldsymbol{R}\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{T}^T\varphi_k = ((\varphi_k^T\boldsymbol{Q}_{k-1}\varphi_k)^{\frac{1}{2}}, 0, \cdots, 0)^T \tag{3}$$

$$c_k = \frac{b_k - \varphi_k^T\boldsymbol{q}_{k-1}}{(\varphi_k^T\boldsymbol{Q}_{k-1}\varphi_k)^{\frac{1}{2}}}. \tag{4}$$

Since the transformed distribution is a isotropic (note that its covariance is an identity matrix), the components are independent to each other and the distribution can be represented as the product of the marginal distributions.

$$p(\boldsymbol{x'}) = \prod_i p_i(x'_i) = \prod_i \frac{1}{\sqrt{2\pi}}\exp(-\frac{{x'_i}^2}{2}) \tag{5}$$

Here, the inequality constraint is $x'_1 \leq c_k$, namely the probability density in the region $x'_1 > c_k$ should be zero by the distribution truncation. Therefore, the original distribution should be normalized so that the definite integral of the normalized distribution over $x'_1 \leq c_k$ is 1.

$$\begin{aligned}\int_{-\infty}^{c_k}\cdots\int_{-\infty}^{\infty} p(\boldsymbol{x'})dx'_1\cdots dx'_n &= \int_{-\infty}^{c_k}\frac{1}{\sqrt{2\pi}}\exp(-{x'_1}^2)dx'_1 \\ &= \frac{1}{2}\left\{1 + \mathrm{erf}\left(\frac{c_k}{\sqrt{2}}\right)\right\}\end{aligned} \tag{6}$$

where $\mathrm{erf}(\cdot)$ denotes the error function:

$$\mathrm{erf}(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})dt. \tag{7}$$

This function is not an analytical function but it can be quite rapidly calculated using a common numerical calculation package. The normalized distribution is as follows:

$$p'(\boldsymbol{x'}) = \alpha\prod_i p_i(x'_i) \tag{8}$$

$$\alpha \equiv \frac{\sqrt{2}}{\sqrt{\pi}(1 + \mathrm{erf}(c_k/\sqrt{2}))}. \tag{9}$$

The mean and covariance of the above distribution are obtained as follows, using the fact that each $x'_i$ is independent:

$$\begin{aligned}\mu_1 = E[x'_1] &= \alpha\int_{-\infty}^{c_k} x'_1\exp(-{x'_1}^2/2)dx'_1 \\ &= -\alpha\exp(-\frac{c_k^2}{2})\end{aligned} \tag{10}$$

$$\begin{aligned}
\mu_i = E[x'_i] &= 0 \ \ (i \neq 1) && (11)\\
\sigma_{11}^2 = E[x'_1 - \mu_1][x'_1 - \mu_1] &= \alpha \int_{-\infty}^{c_k} (x'_1 - \mu_1)^2 \exp(-\frac{{x'}_1^2}{2}) dx'_1 \\
&= 1 - \alpha c_k \exp(-\frac{c_k^2}{2}) - \mu_1^2 && (12)\\
\sigma_{ii}^2 = E[x'_i - \mu_i][x'_i - \mu_i] &= 1 \ \ (i \neq 1) && (13)\\
\sigma_{ij}^2 = E[x'_i - \mu_i][x'_j - \mu_j] &= 0 \ \ (i \neq j). && (14)
\end{aligned}$$

By rewritten the above equations introducing a variable

$$\begin{aligned}
\nu_k &\equiv \mu_1 && (15)\\
&= -\sqrt{\frac{2}{\pi}} \exp(-\frac{c_k^2}{2}) / (1 + \mathrm{erf}(\frac{c_k}{\sqrt{2}})), && (16)
\end{aligned}$$

the mean and covariance are

$$\begin{aligned}
\boldsymbol{\mu}_k &= (\nu_k, 0, \cdots, 0) && (17)\\
\boldsymbol{S}_k &= \mathrm{diag}\{1 + c_k \nu_k - \nu_k^2, 1, \cdots, 1\} && (18)
\end{aligned}$$

where $\mathrm{diag}\{a, b, \cdots\}$ represents a diagonal matrix whose diagonal elements are $a, b, \cdots$. Then the truncated mean and variance are expressed as

$$\begin{aligned}
\boldsymbol{q}_k &= \boldsymbol{T}\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{R}^T\boldsymbol{\mu}_k + \boldsymbol{q}_{k-1} && (19)\\
\boldsymbol{Q}_k &= \boldsymbol{T}\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{R}^T\boldsymbol{S}_k\boldsymbol{R}\boldsymbol{W}^{\frac{1}{2}}\boldsymbol{T}^T && (20)
\end{aligned}$$

by applying inversely the transform Eq.1.